ERRATUM TO: SHELLING-TYPE ORDERINGS OF REGULAR CW-COMPLEXES AND ACYCLIC MATCHINGS OF THE SALVETTI COMPLEX

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1. Overview

In [Del08], a special decomposition of the Salvetti complex of a real hyperplane arrangement is constructed. This construction is based on a choice of an arbitrary linear extension of the arrangement's tope poset. While working on [Müc22], PM noticed that, for the above-mentioned construction to work, the chosen linear extension cannot be arbitrary. The results of [Del08, Section 4] remain valid by work of Lofano and Paolini [LP21], who proved that an appropriate total ordering of the chambers exists for every real hyperplane arrangement. The corresponding claim for non-realizable oriented matroids remains open.

On a technical level, the error in [Del08] is as follows. The claim of [Del08, Notation 4.8] is false in general. This invalidates the proofs of the remainder of the section. However, the statements of [Del08, Thm. 4.13, Cor. 4.15, Lem. 4.18, Prop. 2, Rem. 4.19] remain valid replacing the phrase "every linear extension of [the tope poset]" by "a Euclidean ordering of [the topes]". The definition of Euclidean orderings, and the necessary proofs, were given by Lofano and Paolini in [LP21].

2. Counterexamples

A *tope* of an arrangement $\mathcal{A}$ of hyperplanes in a real vectorspace is the closure of any connected component of $\mathbb{R}^d \setminus \mathcal{A}$. Let $\mathcal{T}(\mathcal{A})$ be the set of topes of $\mathcal{A}$. For $C_1, C_2 \in \mathcal{T}(\mathcal{A})$ let $S(C_1, C_2)$ be the set of all hyperplanes in $\mathcal{A}$ that separate C_1 from C_2.

Now let $B \in \mathcal{T}(\mathcal{A})$. The *tope poset* of $\mathcal{A}$, denoted by $\mathcal{T}(\mathcal{A})_B$, is the set $\mathcal{T}(\mathcal{A})$ endowed with the partial order $\leq_B$ given by $C_1 \leq_B C_2$ if $S(B, C_1) \subseteq S(B, C_2)$.

Let $\mathcal{A}' \subseteq \mathcal{A}$ be a sub-arrangement. Every tope C of $\mathcal{A}$ is contained in a unique tope $\widehat{C}$ of $\mathcal{A}'$. In [Del08, Notation 4.8] it is claimed that $C_1 \leq_B C_2$ if and only if $\widehat{C_1} \leq_{\widehat{B}} \widehat{C_2}$. However, only one implication is correct. For instance, Figure 2 represents the map $C \mapsto \widehat{C}$ for the arrangements of Example 2.1 and there we can see for instance that $\underline{C_{2,7}} \leq_{C_0} C_4$ in $\mathcal{T}(\mathcal{A}')_{C_0}$, but $C_7 \not\leq_{C_0} C_4$ in $\mathcal{T}(\mathcal{A})_{C_0}$. This impairs also the second statement in [Del08, Notation 4.8], i.e., that every linear extension $\dashv$ of the tope poset of $\mathcal{A}$ induces a linear extension $\dashv'$ of the tope poset of $\mathcal{A}' = \mathcal{A} \setminus \{H\}$ by setting $\widehat{C} \dashv \widehat{C'}$ if and only if $C \dashv C'$.

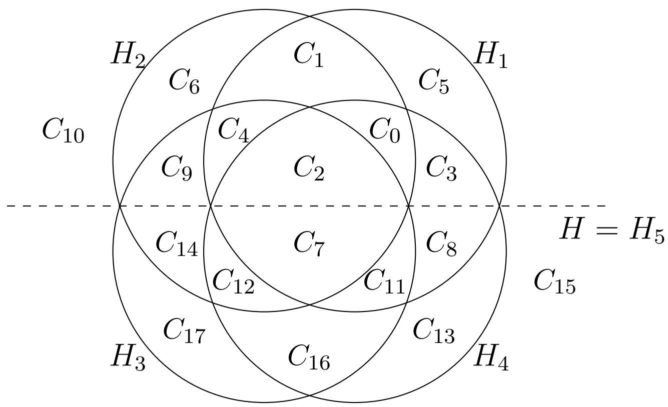

FIGURE 1. A stereographic projection of arrangements $\mathcal{A}$ and $\mathcal{A}' = \mathcal{A} \setminus \{H\} = \mathcal{B}$ and their topes.

Example 2.1. Consider the arrangement $\mathcal{A} = \{\ker(x), \ker(y), \ker(z), \ker(x+y+z), \ker(x+y)\}$ in $\mathbb{R}^3$ and $H = \ker(x+y)$ which is shown in Figure 1. The corresponding map between the tope posets illustrated in Figure 2 does not have a section which is order preserving.

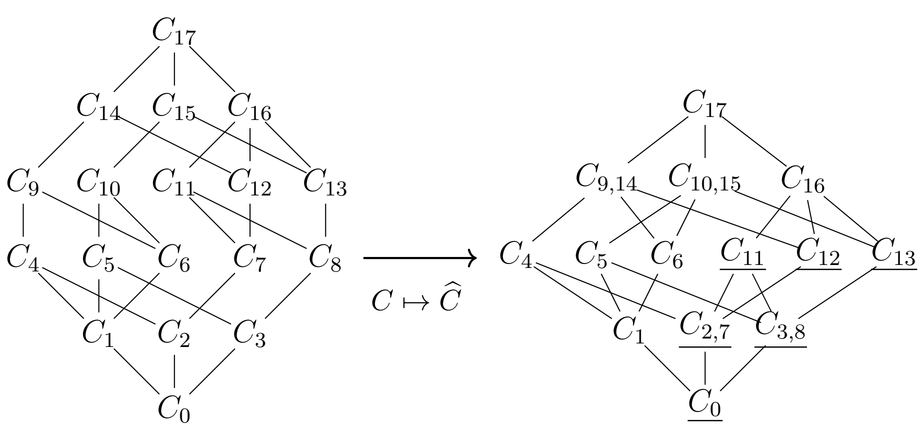

FIGURE 2. The tope posets $\mathcal{T}(\mathcal{A})_{C_0}$ and $\mathcal{T}(\mathcal{A}')_{C_0}$.

The above-mentioned [Del08, Notation 4.8] has its main use in the proof of [Del08, Thm. 4.13], stating that *for any linear extension* $\dashv$ *of* $\mathcal{T}(\mathcal{A})_B$ *and* $C \in \mathcal{T}$ *the set* $J(C) = \{X \in L \mid S(K, C) \cap X \neq \emptyset$ *for all* $K \dashv C\}$ *is a principal order filter of the poset of intersections* $L(\mathcal{A})$ *of* $\mathcal{A}$. This statement, in fact, does not hold for every linear extension (see, e.g., Example 2.2 below). However, Lofano and Paolini proved that every arrangement has at least *some* total orderings of $\mathcal{T}(\mathcal{A})_B$ for which the claim holds, namely what they call *Euclidean orderings*, see [LP21, Def. 5.5]. The same considerations also apply to the results that are derived from [Del08, Thm. 4.13], which remain valid by replacing the phrasing "every linear extension..." with "every Euclidean ordering...".

Example 2.2. Let $\mathcal{A}$ be the arrangement from example 2.1 and $\mathcal{B} = \mathcal{A} \setminus \{H\}$. Consider the order ideal $I = \{C_0, C_{2,7}, C_{3,8}, C_{11}, C_{12}, C_{13}\} \subseteq \mathcal{T}(\mathcal{B})_{C_0}$, illustrated by the underlined elements of the poset on the right hand side of Figure 2. Then there are apparently linear extensions of $\mathcal{T}(\mathcal{B})_{C_0}$ such that all topes of I come first and C_{13} is the last tope in I with respect to the linear extension. Let $\dashv$ be such a linear extension. By looking at Figure 1, we easily see that $H_1 \cap H_2, H_1 \cap H_3 \in L(\mathcal{B})$ are distinct minimal elements of $J(C_{13})$ and consequently $J(C_{13}) \subseteq L(\mathcal{B})$ is not a principal order filter, contradicting [Del08, Thm. 4.13, Lem. 4.18].

Note that the additional conditions on a linear extension of the tope poset that are considered in [Del08, Sec. 5] do not resolve the issue either, since the linear extension described in Example 2.2 can easily be chosen to be of the special form assumed in [Del08, Sec. 5].

3. Open questions

3.1. **Nonrealizable oriented matroids.** It is suggested in [Del08] that the arguments therein are valid for every oriented matroid. Since Euclidean orderings are defined using the metric structure of Euclidean space, they are not defined for tope posets of non-realizable oriented matroids. Therefore, the analogues of the results of [Del08] in the case of nonrealizable oriented matroids remain an open problem.

Question 3.1. Are there always linear orderings of the topes of an oriented matroid such that the order filters $J(C)$ are principal?

A positive answer would prove minimality of the Salvetti complex of arbitrary oriented matroids, as well as lead to a generalization of [Müc22, Thm. 6.4] to general modular flats, not necessarily of corank one – similar to the fibrations presented by Falk and Proudfoot [FP02].

3.2. **Euclidean orderings and linear extensions of tope posets.** In general, Euclidean orderings are not linear extensions of posets of topes.

Question 3.2. Does every real arrangement of linear hyperplanes admit an Euclidean ordering that is also a linear extension of its tope poset (based at some chamber)?

4. Further implications

We point out that the minimality result for toric arrangements obtained in [DD15] remains valid. In that paper, [Del08, Theorem 4.13] is used and stated as Lemma 1.31. As we pointed out, the statement remains valid by choosing an "Euclidean ordering" of the chambers instead of a "linear extension".

Definition 4.1 ([LP21, Def. 5.5 and Rem. 5.4])**.** A total order $\dashv$ of the set $\mathcal{T}$ of topes of a real arrangement is called *Euclidean* if there exists a generic point x_0 such that $\operatorname{dist}(x_0, C) \leq \operatorname{dist}(x_0, C')$ implies $C \dashv C'$. Here "genericity" of x_0 means that every flat of $\mathcal{A}$ has a different distance from x_0.

We now briefly review the instances where total orders on topes are used in [DD15] (they only appear in sections 1, 4 and 5) and we show that they are valid for Euclidean orderings that are not necessarily linear extensions of tope posets.

- Definition 1.15 and Lemma 1.16 remain obviously valid, as they are already stated for any total order of the topes.
- Proposition 1.17 can be be replaced by the analogous statement with respect to Euclidean orderings, as follows.

Proposition 4.2. *Let $\dashv$ be a Euclidean total order of the topes of an arrangement $\mathcal{A}$ with respect to a generic point x_0 and let $\mathcal{A}' \subseteq \mathcal{A}$. For $\widehat{C} \in \mathcal{T}(\mathcal{A}')$ define $\mu(\widehat{C}) := \min_{\dashv}\{C \mid C \subseteq \widehat{C_2}\}$. Define a total order $\dashv'$ on the topes of $\mathcal{A}'$ by imposing that $\widehat{C_1} \dashv' \widehat{C_2}$ if $\mu(\widehat{C_1}) \dashv \mu(\widehat{C_2})$. Then $\dashv'$ is a Euclidean ordering of the topes of $\mathcal{A}'$ with respect to the point x_0.*

Proof. We have to prove that, for $\widehat{C_1}, \widehat{C_2} \in \mathcal{T}(\mathcal{A}')$, $\operatorname{dist}(x_0, \widehat{C_1}) < \operatorname{dist}(x_0, \widehat{C_2})$ implies $\widehat{C_1} \dashv' \widehat{C_2}$. But the fact that $\dashv$ is a Euclidean ordering implies that $\operatorname{dist}(x_0, \widehat{C}_i) = \operatorname{dist}(x_0, \mu(\widehat{C}_i))$ for $i = 1, 2$. Then, $\operatorname{dist}(x_0, \widehat{C_1}) < \operatorname{dist}(x_0, \widehat{C_2})$ implies $\operatorname{dist}(x_0, \mu(\widehat{C_1})) < \operatorname{dist}(x_0, \mu(\widehat{C_2}))$ and this, by definition, entails $\widehat{C_1} \dashv' \widehat{C_2}$. □

- At the beginning of Section 4.1 (top of page 503) we fix a "Euclidean ordering $\prec_0$..." and, accordingly, the total orders in Remark 4.2 are Euclidean.
- Remark 4.9 remains valid since it only depends on the general Lemma 1.16. With this and Lemma 1.31, also Proposition 4.10 and Lemma 4.11, as well as Remark 4.20, remain valid.
- Lemma 5.12 and Lemma 5.15 remain valid since they only use Proposition 4.10, Remark 4.20 and formal manipulations.
- Remark 5.17 goes through invoking [LP21, Lemma 4.12] instead of [Del08, Lemma 4.18], and Lemma 5.18 only uses the bijection proved in Lemma 4.11.
- Section 7 is an appendix on infinite periodic affine arrangements – it also remains valid using "Euclidean orderings" instead of linear extensions.

5. Acknowledgements

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References

[Del08] E. Delucchi, *Shelling-type orderings of regular CW-complexes and acyclic matchings of the Salvetti complex*, Int. Math. Res. Not. **2008** (2008), 39. 1, 2, 3, 4, 5

[DD15] G. d'Antonio and E. Delucchi, *Minimality of toric arrangements*, J. Eur. Math. Soc. **17** (2015), 483–521. 4

[FP02] M. J. Falk and N. J. Proudfoot, *Parallel connections and bundles of arrangements*, Topology Appl. **118** (2002), no. 1-2, 65–83. 3

[LP21] D. Lofano and G. Paolini, *Euclidean matchings and minimality of hyperplane arrangements*, Discrete Math. **344** (2021), no. 3, 22. 1, 3, 4, 5

[Müc22] P. Mücksch, *Modular Flats of oriented matroids and poset quasi-fibrations*, arXiv-preprint (2022). https://arxiv.org/abs/2211.14083 1, 3

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