On formality and combinatorial formality

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Motivation

Theorem [Falk, Randell 1987]
If $\mathcal{A}$ is a hyperplane arrangement in $\mathbb{C}^\ell$ with complement $M(\mathcal{A}) = \mathbb{C}^\ell \setminus \cup_{H\in\mathcal{A}} H$ a $K(\pi,1)$ space, then $\mathcal{A}$ is formal.

Theorem [Yuzvinsky 1993]
If $\mathcal{A}$ is free, then $\mathcal{A}$ is formal.

Remark: Yuzvinsky also demonstrated that formality is not combinatorial, i.e. in general it does not solely depend on the intersection lattice $L(\mathcal{A})$.

Theorem [Brandt, Terao 1994]
If $\mathcal{A}$ is free, then $\mathcal{A}$ is k-formal for all k.

Theorem [Paris 1995]
If $\mathcal{A}$ is a factored arrangement in $\mathbb{R}^3$, then $M(\mathcal{A}\otimes\mathbb{C})$ is a $K(\pi,1)$ space.

Theorem [Amend, Möller, Röhrle 2018]
The class of $K(\pi,1)$-arrangements is not closed under taking restrictions.

Formality [Falk and Randell 1987]

We use the equivalent reformulation by Brandt and Terao.

- Let $\mathcal{A}$ be a hyperplane arrangement in $V \simeq \mathbb{K}^\ell$ with defining linear forms $\alpha_H \in V^*$ for $H \in \mathcal{A}$ such that $H = \ker(\alpha_H)$.
- $F(\mathcal{A}) := \ker\big(\oplus_{H\in\mathcal{A}} \mathbb{K}e_H \to V^*, \quad (a_H \mid H \in \mathcal{A}) \mapsto \sum a_H\alpha_H \big)$ where $(e_H \mid H \in \mathcal{A})$ is the basis for the $\mathbb{K}$-vector space indexed by the hyperplanes in $\mathcal{A}$.
- For $X \in L(\mathcal{A})$ we have a natural inclusion $\oplus_{H\in\mathcal{A}_X} \mathbb{K}e_H \hookrightarrow \oplus_{H\in\mathcal{A}} \mathbb{K}e_H$ which induces an inclusion $i_X : F(\mathcal{A}_X) \hookrightarrow F(\mathcal{A})$.
- $\mathcal{A}$ is called **formal** if $F(\mathcal{A}) = \sum_{X\in L_2(\mathcal{A})} i_X(F(\mathcal{A}_X))$, i.e. if $\pi_2 := \sum_{X\in L_2} i_X : \oplus_{X\in L_2} F(\mathcal{A}_X) \to F(\mathcal{A})$ is surjective.

Example – formality and non-formality

(1) Let $\mathcal{A} = \{H_1,\ldots,H_4\}$ be the arrangement with coefficients of its defining linear forms given by the columns of $A = \left(\begin{smallmatrix} 1&0&0&1\\ 0&1&0&1\\ 0&0&1&0 \end{smallmatrix}\right)$. Then $\dim(F(\mathcal{A})) = \dim(\ker(A)) = 1$ and $X = H_1\cap H_2\cap H_4 \in L_2(\mathcal{A})$ is the only non-simple $\operatorname{codim} 2$ intersection. Hence $F(\mathcal{A}) = F(\mathcal{A}_X)$ and $\mathcal{A}$ is formal.

(2) Consider $\mathcal{B} = \{H_1,\ldots,H_4\}$ with coefficient matrix $B = \left(\begin{smallmatrix} 1&0&0&1\\ 0&1&0&1\\ 0&0&1&1 \end{smallmatrix}\right)$. Again $\dim(F(\mathcal{B})) = \dim(\ker(B)) = 1$ but all $\operatorname{codim} 2$ intersections are simple, i.e. $|\mathcal{B}_X| = 2$ for all $X \in L_2(\mathcal{A})$. Hence $\dim(F(\mathcal{B}_X)) = 0$ for all $X \in L_2(\mathcal{B})$ and $\mathcal{B}$ is not formal.

k-Formality [Brand and Terao 1994]

We can consider higher relation-spaces as follows:

- For $R_3(\mathcal{A}) := \ker(\pi_2)$ we again have natural induced inclusions $i_3(X) : R_3(\mathcal{A}_X) \hookrightarrow R_3(\mathcal{A})$
- Define $\pi_3 := \sum_{X\in L_3} i_3(X) : \oplus_{X\in L_3} R_3(\mathcal{A}_X) \to R_3(\mathcal{A})$
- Call $\mathcal{A}$ **3-formal** if $\mathcal{A}$ is formal and π_3 is surjective.
- We can continue this way by recursively considering $R_k(\mathcal{A}) = \ker(\pi_{k-1})$ and defining $\pi_k : \oplus_{X\in L_k} R_k(\mathcal{A}_X) \to R_k(\mathcal{A})$.
- Call $\mathcal{A}$ **k-formal** if $\mathcal{A}$ is $(k-1)$-formal and π_k is surjective.

Factored arrangements

Let $\pi = (\pi_1,\ldots,\pi_s)$ be a partition of $\mathcal{A}$.

(a) π is called **independent**, provided for any choice $H_i \in \pi_i$ for $1 \leq i \leq s$, the resulting s hyperplanes are linearly independent, i.e. $\operatorname{codim}(H_1\cap\ldots\cap H_s) = s$.

(b) For $X \in L(\mathcal{A})$ let $\pi_X := (\pi_i \cap \mathcal{A}_X \mid \pi_i\cap\mathcal{A}_X \neq \varnothing)$ be the **induced partition** of $\mathcal{A}_X$.

(c) π is a **factorization** of $\mathcal{A}$ provided
 (i) π is independent, and
 (ii) for each $X \in L(\mathcal{A})\setminus\{V\}$, the induced partition π_X admits a block which is a singleton.

If $\mathcal{A}$ admits a factorization, then we also say that $\mathcal{A}$ is **factored**.

Main results in [MMR22]

Theorem
If $\mathcal{A}$ is factored, then it is combinatorially formal.

Theorem
Formality is hereditary, i.e. if $\mathcal{A}$ is formal, then $\mathcal{A}^X$ is formal for any $X \in L(\mathcal{A})$.

Theorem
For $k \geq 3$, the class of k-formal arrangements is not closed under taking restrictions.

Factoredness and formality

- A subset $\mathcal{B} \subseteq \mathcal{A}$ is **line-closed** (in $\mathcal{A}$) if for all $H, H' \in \mathcal{B}$ we have $\mathcal{A}_{H\cap H'} \subseteq \mathcal{B}$.
- For $\mathcal{B} \subseteq \mathcal{A}$ define its **line-closure** $\operatorname{lc}(\mathcal{B}) := \bigcap_{\substack{\mathcal{B}\subseteq\mathcal{C}\subseteq\mathcal{A},\\ \mathcal{C} \text{ is line-closed in } \mathcal{A}}} \mathcal{C}$.

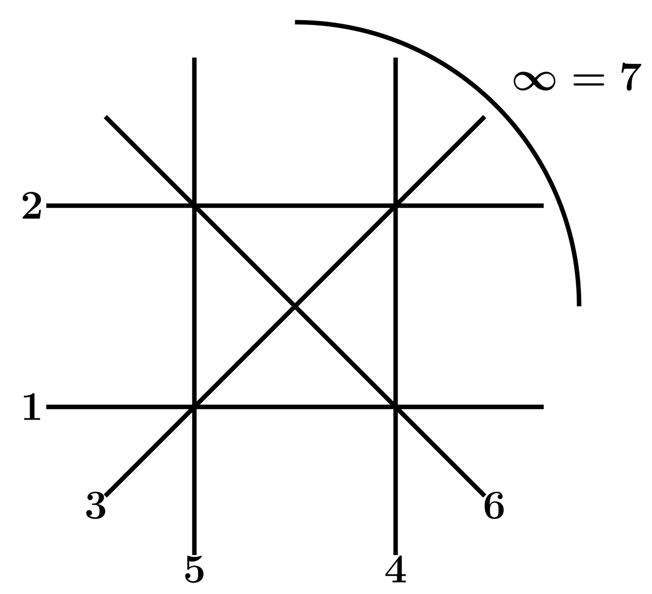
Proposition [Falk 2002]
If there is a subset of hyperplanes $\mathcal{B} \subseteq \mathcal{A}$ such that
- $|\mathcal{B}| = \operatorname{rk}(\mathcal{B}) = \operatorname{rk}(\mathcal{A})$ *and*
- $\operatorname{lc}(\mathcal{B}) = \mathcal{A}$,

then $\mathcal{A}$ is formal.

Key Idea
If $\mathcal{A}$ has a factorization $\pi = (\pi_1,\ldots,\pi_s)$, then there exists a **section** *$S = \{H_1,\ldots,H_s\} \subseteq \mathcal{A}$ of π, i.e. $H_i \in \pi_i$ and $\operatorname{codim}(H_1\cap\ldots\cap H_s) = s$ such that $\operatorname{lc}(S) = \mathcal{A}$.*

Example – factoredness and formality

- Consider $\mathcal{A} = \{H_1,\ldots,H_7\}$ in $\mathbb{R}^3$ with coefficient matrix
$$\begin{pmatrix} 0&0&1&1&1&1&0\\ 1&1&-1&0&0&1&0\\ 0&-1&0&-1&0&-1&1 \end{pmatrix}.$$
- $L_2(\mathcal{A}) = \{127, 135, 234, 146, 256, 457, 36, 37, 67\}$, where we write $i_1\cdots i_k$ for $H_{i_1}\cap\ldots\cap H_{i_k}$.
- $\mathcal{A}$ has a factorization $\pi = (\pi_1,\pi_2,\pi_3) = (\{H_7\},\{H_1,H_2,H_3\},\{H_4,H_5,H_6\})$.



- For the section $S' = \{H_7 \in \pi_1, H_3 \in \pi_2, H_6 \in \pi_3\}$ we only have $\operatorname{lc}(S') = S'$, i.e. S' is line-closed in $\mathcal{A}$.
- But for $S = \{H_7 \in \pi_1, H_1 \in \pi_2, H_4 \in \pi_3\}$ we have $\operatorname{lc}(S) = \mathcal{A}$.
- $\mathcal{A}$ is also simplicial and by one of our results [MMR22, Prop. 2.8] the walls of every single chamber form an lc-basis, e.g. $\{H_3, H_4, H_6\}$.

Example – k-formality is not hereditary

The following example was found by means of a modified "greedy-algorithm" based on recent ideas presented by Cuntz.

- Consider the 5-arrangement $\mathcal{A} = \{H_1,\ldots,H_{11}\}$ in $\mathbb{K}^5$ with coefficient matrix $\left(\begin{smallmatrix} 1&0&0&0&0&1&1&1&1&1&1\\ 0&1&0&0&0&0&0&1&1&1&1\\ 0&0&1&0&0&1&1&0&0&1&1\\ 0&0&0&1&0&0&0&1&1&0&1\\ 0&0&0&0&1&0&1&0&1&1&1 \end{smallmatrix}\right)$.
- One can check, that the underlying matroid of $\mathcal{A}$ is *regular*, i.e. is realizable over any field $\mathbb{K}$.
- $\mathcal{A}$ is k-formal for all k.
- Yet, the restriction $\mathcal{A}^{H_2}$ in $\mathbb{K}^4$ with coefficient matrix $\left(\begin{smallmatrix} 1&0&0&0&1&1&1&1&1\\ 0&1&0&0&1&1&0&0&1\\ 0&0&1&0&0&0&1&1&1\\ 0&0&0&1&0&1&0&1&1 \end{smallmatrix}\right)$ is $(2$-$)$formal but not 3-formal.

Questions

- Are factored arrangements k-formal for all k?
- Is k-formality (for $k \geq 3$) necessary for complex arrangements to have aspherical complements?

Main Reference

[MMR22] T. Möller, P. Mücksch, and G. Röhrle, *On formality and combinatorial formality for hyperplane arrangements*, arXiv:2202.09104 (2022) https://arxiv.org/abs/2202.09104

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