Milnor fibrations and oriented matroids

Paul Mücksch[1]
(joint work with Masahiko Yoshinaga[2])
[1] Technical University Berlin, Germany; [2] Osaka University, Japan

Motivation

An important instance of a most non-isolated singularity is a **hyperplane arrangement** $\mathcal{A}$ in $V = \mathbb{C}^\ell$.

- Let $\alpha_H \in (\mathbb{C}^\ell)^*$ $(H = \ker(\alpha_H) \in \mathcal{A})$ be defining linear forms for $\mathcal{A}$,
- $Q = \prod_{H\in\mathcal{A}} \alpha_H \in \mathbb{C}[x_1, \dots, x_\ell]$ the corresponding **defining polynomial** of $\mathcal{A}$,
- $\mathfrak{X} = V \setminus \bigcup_{H\in\mathcal{A}} H$ the arrangement **complement**.

The **Milnor fibration** of $\mathcal{A}$ is

$$Q|_{\mathfrak{X}} : \mathfrak{X} \to \mathbb{C}^\times, v \mapsto Q(v),$$

and its **Milnor fiber** we denote by $\mathfrak{F} := Q^{-1}(1)$.

For the complement $\mathfrak{X}$ of a complexified real arrangement, the foundational work of Salvetti provided a combinatorial model in the form of the **Salvetti complex**, a finite regular CW complex whose homotopy type depends only on the oriented matroid of the arrangement (see below).

In contrast, a concrete model for the homotopy type of $\mathfrak{F}$ is available only in the special cases of real reflection arrangements, thanks to Brady, Falk, and Watt and the generic case due to Orlik and Randell.

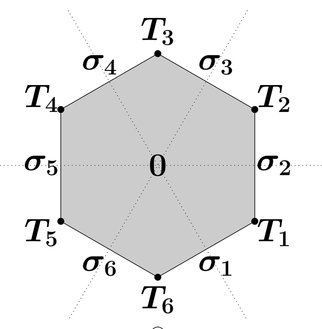
The Salvetti complex

Let $\mathcal{L}$ be the covectors poset of an oriented matroid $\mathcal{M}$ and $\mathcal{T}$ its topes, i.e. maximal elements of $\mathcal{L}$. Then the (face poest of the) **Salvetti complex** $\mathcal{S}$ of $\mathcal{M}$ is defined as

$$\mathcal{S} := \{(\sigma, T) \mid T \in \mathcal{T} \text{ and } \sigma \in \mathcal{L}_{\leq T}\} \subseteq \mathcal{L} \times \mathcal{T},$$

with partial order

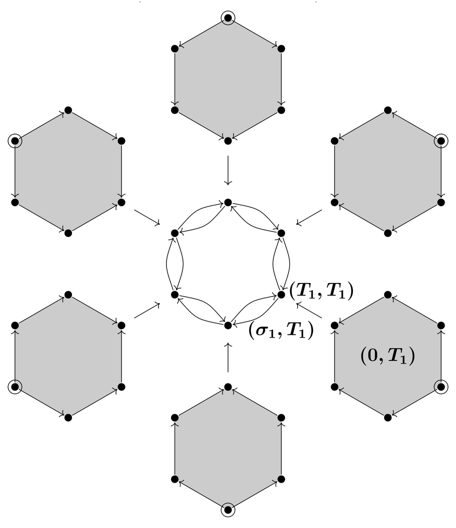
$$(\sigma, T) \leq_{\mathcal{S}} (\tau, R) :\iff \sigma \geq_{\mathcal{L}} \tau \text{ and } \sigma \circ R = T.$$



Theorem [Salvetti 1987].

The Salvetti complex of $\mathcal{M}(\mathcal{A})$ is homotopy equivalent to the complement of the complexified arrangement:

$$|\mathcal{S}| \cong V \otimes \mathbb{C} \setminus \left(\bigcup_{H\in\mathcal{A}} H \otimes \mathbb{C} \right).$$



The Salvetti complex $\mathcal{S}$ has

- one vertex for each of chamber (or tope),
- two edges connecting each pair of adjacent chambers, represented by two arrows pointing in opposite directions,
- and one k-cell for each $\sigma \in \mathcal{L}^\vee$ of codimension k and tope T adjacent to σ, PL homeomorphic to $|\mathcal{L}^\vee_{\leq\sigma}|$, whose boundary cells are identified with $k-1$ cells with the same edges oriented away from T.

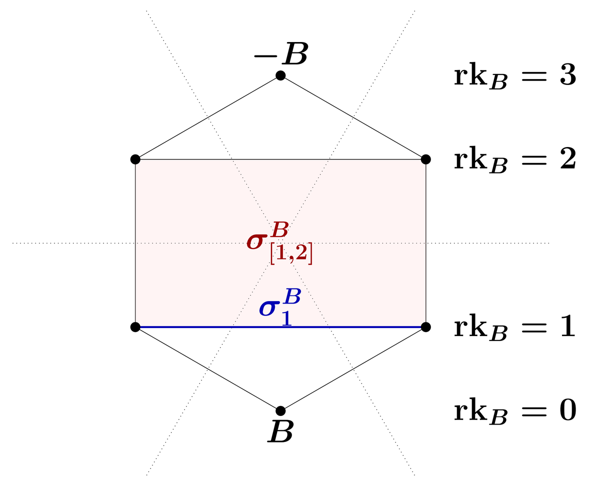
Definition – The tope rank subdivision

Let $B \in \mathcal{T}$ be a tope. We define a partial order on $\mathcal{T}$ by $R \leq_B T :\iff S(B, R) \subseteq S(B, T)$, where $S(B, R)$ denotes the set of (pseudo-)hyperplanes separating B and R. The resulting ranked poset $\mathcal{T}_B = (\mathcal{T}, \leq_B)$ with rank function $\mathrm{rk}_B(T) := |S(B, T)|$ is called the **tope poset** with respect to B.

Let $\sigma \in \mathcal{L}^\vee$ be a cell in the dual covector complex and $B \in \mathcal{T}$ a tope. Recall that we set $\mathcal{T}(\sigma) := \mathcal{T} \cap \mathcal{L}^\vee_{\leq\sigma}$ which can be identified with $\mathrm{vert}(\sigma)$ and define

- $\sigma_k^B := \{T \in \mathcal{T}(\sigma) \mid \mathrm{rk}_B(T) = k\}$,
- $\sigma_{[k,k+1]}^B := \sigma_k^B \cup \sigma_{k+1}^B$,
- define the **$(B$-)rank subdivision** of σ as:

$$\mathrm{rk}_B \,\mathrm{sd}(\sigma) := \{\sigma_k^B \mid k \in \mathrm{rk}_B(\mathcal{T}(\sigma))\} \cup \{\sigma_{[k,k+1]}^B \mid k \in \mathrm{rk}_B(\mathcal{T}(\sigma)) \setminus \{\mathrm{rk}_B(\sigma \circ (-B))\}\}.$$



Then the **$(B$-)rank subdivision** of $\mathcal{L}^\vee$ is the poset defined by: $\mathrm{rk}_B \,\mathrm{sd}\, \mathcal{L}^\vee := \bigcup_{\sigma\in\mathcal{L}^\vee} \mathrm{rk}_B \,\mathrm{sd}(\sigma) \subseteq 2^{\mathcal{T}}$ with partial order by inclusion.

We further have a poset map to the original complex:

$$p_B := p|_{\mathrm{rk}_B \,\mathrm{sd}\, \mathcal{L}^\vee} : \mathrm{rk}_B \,\mathrm{sd}\, \mathcal{L}^\vee \to \mathcal{L}^\vee, \mathcal{T} \supseteq \mathfrak{a} \mapsto \min\{\sigma \in \mathcal{L}^\vee \mid \mathfrak{a} \subseteq \mathcal{T}(\sigma)\}.$$

The *tope-rank subdivision* of a cell $(\sigma, T) \in \mathcal{S}$ is defined as: $\mathrm{rk}\,\mathrm{sd}(\sigma, T) := \{(\mathfrak{a}, T) \mid \mathfrak{a} \in \mathrm{rk}_T \,\mathrm{sd}(\sigma)\}$, and the **tope-rank subdivision** of $\mathcal{S}$ is defined by $\mathrm{rksd}\mathcal{S} := \bigcup_{x\in\mathcal{S}} \mathrm{rk}\,\mathrm{sd}(x)$, with partial order given by $(\mathfrak{a}, T) \leq (\mathfrak{b}, R) :\iff \mathfrak{a} \subseteq \mathfrak{b}$ and $p_T(\mathfrak{a}) \circ R = T$. We have a poset map $\widetilde{p} : \mathrm{rksd}\mathcal{S} \to \mathcal{S}, (\mathfrak{a}, T) \mapsto (p_T(\mathfrak{a}), T)$.

Combinatorial models of fibrations

For a poset map $f : P \to Q$ we write $(f \downarrow q) := f^{-1}(Q_{\leq q})$ for **poset fibers** of f $(q \in Q)$.

Theorem [Quillen's Theorem B for posets 1973].

If for all $a \leq b$ $(a, b \in Q)$ the inclusion $(f \downarrow a) \hookrightarrow (f \downarrow b)$ is a homotopy equivalence, the homotopy fiber $\mathrm{HoFib}(|\Delta(f)|, a)$ is homotopy equivalent to $|\Delta(f \downarrow a)|$.

If for all $a \leq b$ $(a, b \in Q)$ the inclusion $(f \downarrow a) \hookrightarrow (f \downarrow b)$ is a homotopy equivalence, then f is called a **poset quasi-fibration**. Let $\varphi : X \to Y$ be a topological fibration. Then we say that f is a **combinatorial model** for φ if f is a poset quasi-fibration and a (homotopy) commutative diagram:

$$\begin{tikzcd} {|\Delta(P)|} \arrow[r, "|\Delta(f)|"] \arrow[d, "\simeq"'] & {|\Delta(Q)|} \arrow[d, "\simeq"] \\ X \arrow[r, "\varphi"] & Y, \end{tikzcd}$$

where the vertical maps are homotopy equivalences.

Definition – combinatorial Milnor fibration

Let $\mathcal{C}$ denote the Salvetti complex of the rank 1 oriented matroid with covectors $\{+, -, 0\}$, i.e. the face poset of $\mathcal{C}$ is given by the set $\{(+,+), (0,+), (-,-), (0,-)\}$, a regular cell decomposition of the circle S^1.

Define the map $Q : \mathcal{T} \to \{+, -\}, T \mapsto \prod_{e\in E} T_e$, and the poset map $\widetilde{Q} : \mathrm{rksd}\mathcal{S} \to \mathcal{C}$ by:

$$\widetilde{Q}((\sigma_k^T, T)) := \begin{cases} (+,+) & \text{if } Q(\sigma_k^T) = \{+\}, \\ (-,-) & \text{if } Q(\sigma_k^T) = \{-\}, \end{cases}$$

and

$$\widetilde{Q}((\sigma_{[k,k+1]}^T, T)) := \begin{cases} (0,+) & \text{if } Q(\sigma_k^T) = \{+\}, \\ (0,-) & \text{if } Q(\sigma_k^T) = \{-\}. \end{cases}$$

We define the **(combinatorial) Milnor fiber** of $\mathcal{M}$ by $\widetilde{\mathfrak{F}}(\mathcal{M}) := \widetilde{Q}^{-1}((+,+))$.

Main results in [MY25]

Theorem 1.

The poset $\mathrm{rksd}\mathcal{S}$ is the face poset of a regular cell complex PL-homeomorphic to $\mathcal{S}$ and if $\mathcal{S} = \mathcal{S}(\mathcal{A})$ is the Salvetti-complex of a real arrangement $\mathcal{A}$, then $|\mathrm{rksd}\mathcal{S}|$ is homotopy equivalent to the complexified complement $\mathfrak{X}(\mathcal{A})$.

Theorem 2.

The map $\widetilde{Q} : \mathrm{rksd}\mathcal{S} \to \mathcal{C}$ is a poset quasi-fibration.

Theorem 3.

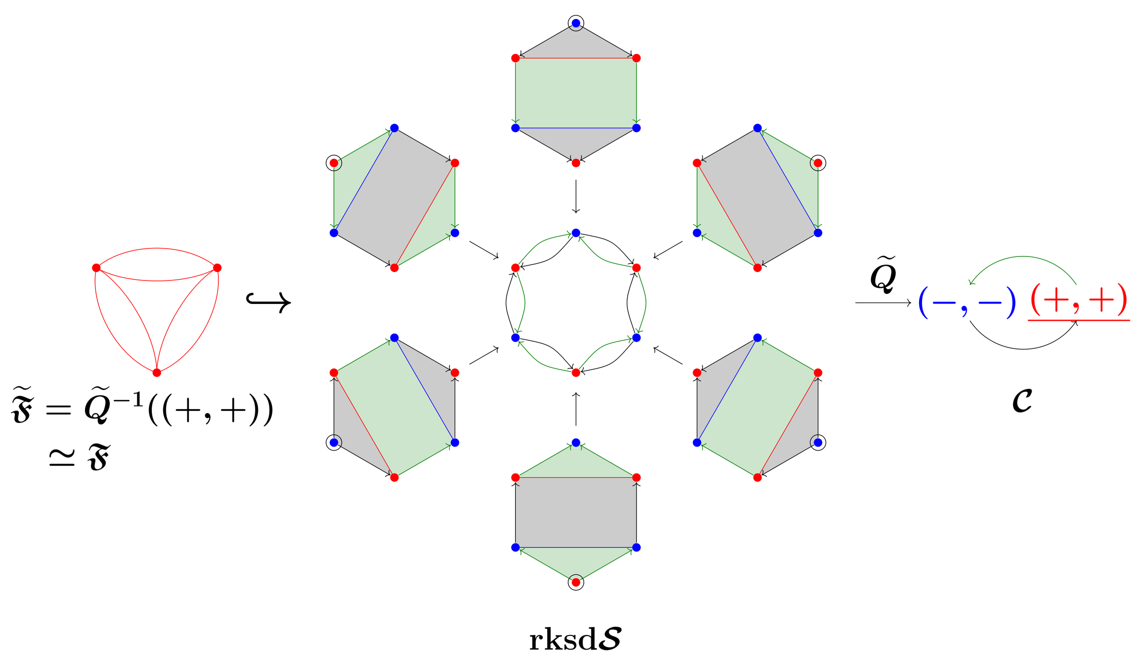
There is a (homotopy) commutative square

$$\begin{tikzcd} {|\mathrm{rksd}\mathcal{S}(\mathcal{A})|} \arrow[r, "|\widetilde{Q}|"] \arrow[d, "\simeq"'] & {|\mathcal{C}|} \arrow[d, "\simeq"] \\ \mathfrak{X}(\mathcal{A}) \arrow[r, "Q"] & \mathbb{C}^\times, \end{tikzcd}$$

where the vertical maps are homotopy equivalences, i.e. $\widetilde{Q}$ is a combinatorial model for the Milnor fibration of $\mathcal{A}$.

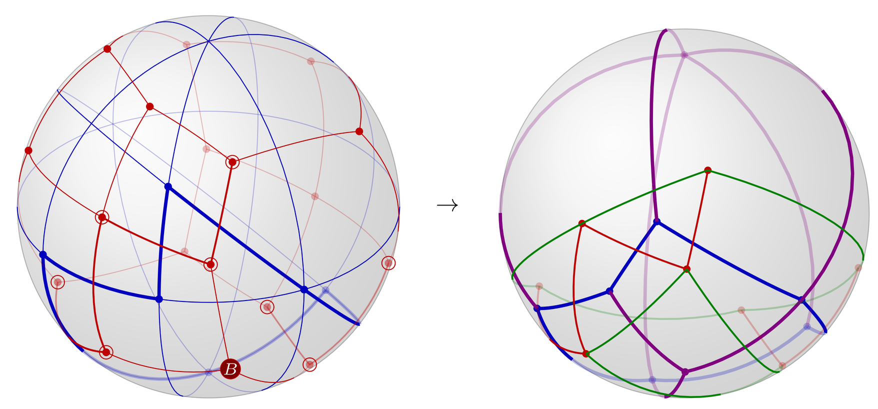
Theorem 4.

The combinatorial Milnor fiber $\widetilde{\mathfrak{F}}$ is homotopy equivalent to the geometric Milnor fiber $\mathfrak{F}$.

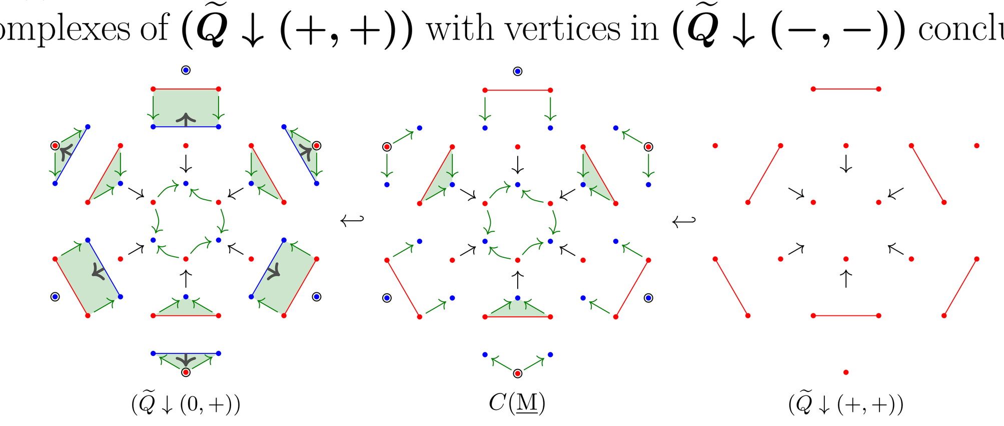


About the proofs

Theorem 1: consider the covector complex $\mathcal{L}$ together with its dual $\mathcal{L}^\vee$.
We construct a new complex $\Sigma_{[k,k+1]}$ with respect to $B \in \mathcal{T}$ from a certain subcomplex of $\mathcal{L} \setminus \{0\}$ which is the face poset of a regular cell decomposition of the PL $(d-1)$-sphere.
Then, by construction, $\mathcal{F}(\Sigma^\vee_{[k,k+1]})$ is isomorphic to $\mathrm{rk}_B \,\mathrm{sd}\, \mathcal{L}^\vee_{<0^B_{[k,k+1]}}$ which is also a PL $(d-1)$-sphere.



Theorem 2: we use Forman's **Discrete Morse Theory**. We construct an acyclic matching $\underline{\mathrm{M}}$ on $(\widetilde{Q} \downarrow (0,+))$ giving a first homotopy equivalence. Then, "pushing in" remaining cones over contractible subcomplexes of $(\widetilde{Q} \downarrow (+,+))$ with vertices in $(\widetilde{Q} \downarrow (-,-))$ concludes our argument:



Theorem 3: assume that $|\mathcal{A}| = n$ and let $\mathcal{B}_n$ be the Boolean arrangement of rank n. We split up the diagram into smaller parts as follows, each of which commutes (up to homotopy):

$$\begin{tikzcd} {|\mathrm{rksd}\mathcal{S}(\mathcal{A})|} \arrow[r, hook, "i'"] \arrow[d, "|\widetilde{p}_{\mathcal{A}}|"] \arrow[dd, bend right] \arrow[rr, bend left, "|\widetilde{Q}(\mathcal{A})|"] & {|\mathrm{rksd}\mathcal{S}(\mathcal{B}_n)|} \arrow[r, "|\widetilde{Q}(\mathcal{B}_n)|"] \arrow[d, "|\widetilde{p}_{\mathcal{B}_n}|"'] \arrow[dd, bend left, "\widetilde{\varphi}"] & {|\mathcal{C}|} \arrow[dd, "\simeq"] \\ {|\mathcal{S}(\mathcal{A})|} \arrow[r, hook, "i"] \arrow[d, "\simeq"] & {|\mathcal{S}(\mathcal{B}_n)|} \arrow[d, "\simeq"'] & \\ \mathfrak{X}(\mathcal{A}) \arrow[r, hook] \arrow[rr, bend right, "Q(\mathcal{A})"'] & (\mathbb{C}^\times)^n \arrow[r, "Q(\mathcal{B}_n)"] & \mathbb{C}^\times. \end{tikzcd}$$

Main Reference

[MY25] P. Mücksch, and M. Yoshinaga, **Milnor fibrations and oriented matroid**, arXiv:2508.15331 (2025) preprint.

Contact Information

- Web: https://paulmuecksch.github.io/
- Email: paul.muecksch+uni@gmail.com